REDUCIBILITY AND DIAGONAL EQUALITY STRATA FOR FINITE JACOBI PENCILS

MATTHEW FAUST

Abstract. We study reducibility of the characteristic polynomials of a class of finite Jacobi pencils. We classify the diagonal equality patterns for which generic connected couplings yield reducibility and use this classification to prove that for $n \geq 4$, the reducible locus has codimension at least two in the connected parameter space. For $n = 2m + 1$, we also prove that for every fixed connected coupling vector, the locus of diagonal potentials that admit a constant branch has codimension m.

1. Introduction

We consider reducibility of the characteristic polynomial of Jacobi matrices of the form

$$J_n(w) = \begin{pmatrix} a_1 & wb_1 & 0 & \cdots & 0 \\ wb_1 & a_2 & wb_2 & \ddots & \vdots \\ 0 & wb_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & wb_{n-1} \\ 0 & \cdots & 0 & wb_{n-1} & a_n \end{pmatrix}.$$

We use the sign convention

$$\chi_n(w, \lambda) = \det(J_n(w) + \lambda I).$$

Since w only appears in the off-diagonal entries, χ_n is even in w. We write

$$\chi_n(w, \lambda) = P_n(t, \lambda), \qquad t = w^2,$$

where

$$P_0 = 1, \qquad P_1 = \lambda + a_1, \qquad P_k = (\lambda + a_k)P_{k-1} - tc_{k-1}P_{k-2},$$

with $c_i = b_i^2$.

A recent preprint of Shapiro [Sha26b] studies when $\chi_n(w, \lambda)$ is reducible. Shapiro first proves that, for fixed pairwise-distinct diagonal entries $a_1, \ldots, a_n$, the polynomial χ_n is irreducible for all $b = (b_1, \ldots, b_{n-1})$ outside a proper algebraic subset [Sha26b, Theorem 2.1]. The same paper then proposes that, for

2020 *Mathematics Subject Classification.* 47B36, 15A22, 12D05.

Key words and phrases. determinantal polynomials, Jacobi pencils, reducibility.

$n \geq 4$, the only codimension-one components of the reducible locus are the cut hyperplanes $b_i = 0$ (see [Sha26b, Conjecture 4.3]).

In this paper, we prove [Sha26b, Conjecture 4.3] (Theorem 7.1). In addition to this, for any fixed connected coupling, we give the dimension of the class of odd length potentials that admit constant branches (Theorem 4.7). We also characterize generic reducibility for arbitrary fixed diagonal data: in odd length, the odd-indexed diagonal entries must all coincide, while in even length, each parity class must be constant (Theorem 6.1 and Corollary 6.4). We note that the methods used are close in spirit to those developed for studying the reducibility of dispersion polynomials [Liu22, FLG25, FLM22, FL26]. Indeed, one can view $\chi_n(w, \lambda)$ as the dispersion polynomial of the discrete operator $J_n(w) = A + w\Delta_b$ on an n vertex path graph, where A is a diagonal matrix with entries a_i (the potential) and Δ_b is a weighted graph adjacency operator with couplings b_i.

Organization: In Section 2, we collect elementary reductions. In Section 3, we use Newton polytopes to constrain factors. In Section 4, we classify constant branches. In Section 5, we prove the codimension bound for fixed pairwise-distinct diagonal data. In Section 6, we classify generic reducibility for arbitrary fixed diagonal data, both for P_n and for χ_n. Finally, in Section 7, we prove the codimension bound on the full connected parameter space.

Remark 1.1. This manuscript was prepared in response to the first version of Shapiro's preprint [Sha26b]. A second version appeared [Sha26a] while this manuscript was in preparation and contains some overlapping preliminary results. To our knowledge, the main results (Theorem 4.7, Theorem 6.1, Corollary 6.4, and Theorem 7.1) are new. We also note that these results do not require the diagonal entries of J_n to be pairwise distinct, which is the primary setting of [Sha26a]. For completeness, Sections 2, 3, and 4 retain some results that overlap with the revised preprint.

2. Basic Reductions

We begin by collecting notation and making some elementary reductions. We call $\chi_n(w, \lambda)$ the spectral polynomial, $P_n(t, \lambda)$ the reduced continuant, and write

$$A_i = \lambda + a_i.$$

For a contiguous block $[r, s]$, we write $P_{r,s}(t, \lambda)$ for the corresponding reduced continuant. That is, $P_{r,s}(t, \lambda) = \det(J_{r,s}(\sqrt{t}) + \lambda I)$, where

$$J_{r,s}(w) = \begin{pmatrix} a_r & wb_r & 0 & \cdots & 0 \\ wb_r & a_{r+1} & wb_{r+1} & \ddots & \vdots \\ 0 & wb_{r+1} & a_{r+2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & wb_{s-1} \\ 0 & \cdots & 0 & wb_{s-1} & a_s \end{pmatrix}.$$

Recall that $c_i = b_i^2$, we say the chain is connected if

$$c_1 \cdots c_{n-1} \neq 0.$$

If $c_i = 0$, the chain cuts and the characteristic polynomial factors as the product of the two subchain characteristic polynomials. We write

$$T_c = (\mathbb{C}^*)^{n-1}$$

for the connected coupling torus with coordinates $c_1, \ldots, c_{n-1}$. Unless explicitly stated otherwise, we work over T_c.

Lemma 2.1. *The affine change $\lambda \mapsto \lambda + s$ preserves reducibility in both $\mathbb{C}[w, \lambda]$ and $\mathbb{C}[t, \lambda]$. In particular, shifting all diagonal entries by the same scalar does not change reducibility.*

Proof. This is just applying a polynomial-ring automorphism. □

Lemma 2.2. *For a block $[r, s]$,*

$$P_{r,s}(t, \lambda) = \sum_M (-t)^{|M|} \left(\prod_{e \in M} c_e \right) \prod_{j \in [r,s] \setminus V(M)} A_j,$$

where the sum is over matchings of the path $[r, s]$. Consequently, if $t^k \lambda^\ell$ appears in P_n, then

$$2k + \ell \leq n.$$

Proof. The matching formula follows directly from the Leibniz expansion of the determinant. The support inequalities are immediate, since a matching of size k removes $2k$ vertices from the product of the A_i. □

Lemma 2.3. *Let $F(t, \lambda)$ be a monic factor of $P_n(t, \lambda)$ of λ-degree d. Write*

$$F(t, \lambda) = \lambda^d + f_1(t)\lambda^{d-1} + \cdots + f_d(t).$$

Then $\deg_t f_j \leq \left\lfloor \frac{j}{2} \right\rfloor$. Equivalently, if

$$F(t, \lambda) = F_0(\lambda) + tF_1(\lambda) + t^2 F_2(\lambda) + \cdots,$$

then $\deg_\lambda F_k \leq d - 2k$. In particular, monic linear factors are independent of t.

Proof. Let $P_n = FG$, where G is monic of λ-degree $n - d$. The Newton polytope of a product is the Minkowski sum of the Newton polytopes of the factors, and P_n is supported in the halfspace $2k + \ell \leq n$ by Lemma 2.2. Since G is monic, $(0, n - d) \in N(G)$. Thus, for every $(k, \ell) \in N(F)$, the point $(k, \ell + n - d)$ lies in $N(P_n)$, and hence

$$2k + \ell + n - d \leq n.$$

Therefore, $2k + \ell \leq d$. □

Lemma 2.4. *Assume the a_i are pairwise distinct. If*

$$\chi_n(w,\lambda) = F(w,\lambda)G(w,\lambda)$$

with F, G monic in λ, then F and G are even in w. Hence, every factorization of $\chi_n(w,\lambda)$ descends to a factorization of $P_n(t,\lambda)$, and conversely.

Moreover, after setting $w = 0$, a monic factor determines a unique subset $S \subset \{1,\dots,n\}$ by $F(0,\lambda) = \prod_{i\in S} A_i$.

Proof. At $w = 0$, $\chi_n(0,\lambda) = \prod_{i=1}^n A_i$ is squarefree. Thus, the specialization of a monic factor is the product of a unique subset of the A_i. From this, together with the involution $w \mapsto -w$ preserving χ_n, we conclude $F(-w,\lambda) = F(w,\lambda)$. □

On the pairwise-distinct diagonal stratum, Lemma 2.4 allows us to attach a unique subset $S \subset [n]$ to any monic factor by specializing $w = 0$. Following [Sha26b], we call this subset the Hensel subset of the factor. Equivalently, a factorization has Hensel subset S if one of its monic factors specializes to $\prod_{i\in S} A_i$ at $w = 0$, or to the same product after setting $t = 0$ for P_n.

Lemma 2.5. *Let $P(t,\lambda) \in \mathbb{C}[t,\lambda]$ be monic in λ and irreducible in $\mathbb{C}[t,\lambda]$, and set*

$$X(w,\lambda) = P(w^2,\lambda).$$

If X is reducible, then its irreducible factors are permuted by $w \mapsto -w$. The product over any orbit descends to $\mathbb{C}[t,\lambda]$, and thus is a factor of P. Therefore, if P is irreducible, then either X is irreducible or $X(w,\lambda) = f(w,\lambda)f(-w,\lambda)$.

Proof. The involution $w \mapsto -w$ acts on the irreducible factors of X. The product over an orbit is invariant under this involution. Since P is irreducible, there can be only one orbit unless X splits as a conjugate pair.

Indeed, suppose that f is an irreducible factor of X. If $f(w,\lambda)$ and $f(-w,\lambda)$ are distinct factors of X, then $f(\sqrt{t},\lambda)f(-\sqrt{t},\lambda)$ divides P and thus equals P as P is irreducible. Otherwise, $f(w,\lambda)$ is fixed by $w \mapsto -w$ and so $f(\sqrt{t},\lambda)$ is a factor of P, and therefore equal to P; and so $f = X$. □

Corollary 2.6. *If P_n is irreducible and n is odd, then χ_n is irreducible. If P_n is irreducible and n is even, then either χ_n is irreducible or $\chi_n(w,\lambda) = f(w,\lambda)f(-w,\lambda)$, where both factors have λ-degree $n/2$.*

Proof. Apply Lemma 2.5. In odd λ-degree, a conjugate pair factorization would have to split the degree evenly, which is impossible. □

3. Newton Polytope Constraints on Factors

In this section Newton polytopes are taken in the (t,λ)-exponent coordinates.

Theorem 3.1. *Suppose that no $c_i = b_i^2$ is zero. If $n = 2m$, then for generic $s \in \mathbb{C}$,*

$$\mathcal{N}(P_n(t,\lambda+s)) = conv\{(0,0),(m,0),(0,2m)\}.$$

Proof. For $n = 2m$, the point $(0, 2m)$ comes from λ^{2m}, and $(m, 0)$ comes from the permutation

$$(1,2), (3,4), \ldots, (2m-1, 2m),$$

in the Leibniz expansion of the determinant whose coefficient is

$$(-1)^m c_1 c_3 \cdots c_{2m-1} \neq 0.$$

A generic shift $\lambda \mapsto \lambda + s$ makes the constant term nonzero, giving the claimed triangle. Since a triangle is only homothetically decomposable [FLG25], any Newton polytope of a factor is homothetic to this triangle, up to translation and the degenerate point case. $\square$

Corollary 3.2. *Under the assumptions of Theorem 3.1: When n is even, all factors of $P_n(t, \lambda)$ have even degree in λ.*

4. On Constant Branches

Lemma 4.1. *We may write:*

$$\frac{P_{1,j-1}}{P_{1,j}} = \cfrac{1}{L_1 - \cfrac{tc_{j-1}}{L_2 - \cfrac{tc_{j-2}}{\ddots - \cfrac{tc_1}{L_j}}}}$$

Where $L_i = \lambda + a_{j-i+1}$.

Proof. This is obtained by repeatedly dividing the recurrence

$$P_{1,k} = (\lambda + a_k) P_{1,k-1} - tc_{k-1} P_{1,k-2}$$

by $P_{1,k-1}$, starting from $k = j$ and decrementing. $\square$

Lemma 4.2. *Let $M(t)$ and $N(t)$ be two rational functions of the form*

$$M = \cfrac{1}{L_1 - \cfrac{tu_2}{L_2 - \cfrac{tu_3}{\ddots - \cfrac{tu_l}{L_l}}}}, \qquad N = \cfrac{1}{R_1 - \cfrac{tv_2}{R_2 - \cfrac{tv_3}{\ddots - \cfrac{tv_r}{R_r}}}},$$

where L_i, R_i, u_i, v_i are all nonzero. Suppose that $\alpha M + \beta N = 0$ as a rational function, where $\alpha, \beta \neq 0$, then $l = r$. Moreover, for this equality to be satisfied, we must have that $\beta = -\alpha \frac{R_1}{L_1}$ and $v_s = u_s \frac{R_{s-1} R_s}{L_{s-1} L_s}$ for $s = 2, \ldots, l$.

Proof. Define

$$M_i(t) = \cfrac{1}{L_i - \cfrac{tu_{i+1}}{L_{i+1} - \cfrac{tu_{i+2}}{\ddots - \cfrac{tu_l}{L_l}}}}, \qquad N_j(t) = \cfrac{1}{R_j - \cfrac{tv_{j+1}}{R_{j+1} - \cfrac{tv_{j+2}}{\ddots - \cfrac{tv_r}{R_r}}}}.$$

First, notice that letting $t = 0$ gives $\beta = -\alpha \frac{R_1}{L_1}$.

Substituting this back in gives us $\alpha M - \alpha \frac{R_1}{L_1} N = 0$, and so we have that $M = \frac{R_1}{L_1} N$. Inverting both sides gives us $\frac{1}{M} = \frac{L_1}{R_1}\frac{1}{N}$, where $\frac{1}{M} = L_1 - tu_2 M_2(t)$ and $\frac{1}{N} = R_1 - tv_2 N_2(t)$.

Thus $t(u_2 M_2 - \frac{L_1}{R_1} v_2 N_2) = 0$, so factoring out t forces $u_2 M_2 - \frac{L_1}{R_1} v_2 N_2 = 0$. Note that this is the same type of relation that we started with. Setting $t = 0$ again gives us that $v_2 = u_2 \frac{R_1 R_2}{L_1 L_2}$,

Plugging this in for v_2 and repeating the same process, we obtain $u_3 M_3 - \frac{L_2}{R_2} v_3 N_3 = 0$, or more generally, we see that $u_s M_s - \frac{L_{s-1}}{R_{s-1}} v_s N_s = 0$, and so $v_s = u_s \frac{R_{s-1} R_s}{L_{s-1} L_s}$.

After interchanging M and N, it is enough to rule out $l < r$. In that case $M_l = \frac{1}{L_l}$ and $N_l = \frac{1}{R_l - tv_{l+1} N_{l+1}}$ are linearly dependent. However, this would mean there is a nonconstant function that is given by a constant function. ☐

Theorem 4.3. *Suppose that the a_i are pairwise distinct and $c_i \neq 0$. Let $n = 2m + 1 \geq 3$. If $P_n(t, \lambda)$ has a factor of the form $\lambda + \alpha$, then $\alpha = a_{m+1}$.*

Moreover, $\lambda + a_{m+1}$ divides $P_n(t, \lambda)$ if and only if the c_i satisfy the following relations:

$$c_{m+1} = -c_m \frac{a_{m+2} - a_{m+1}}{a_m - a_{m+1}},$$

and

$$c_{m+s} = c_{m+1-s} \frac{(a_{m+s} - a_{m+1})(a_{m+s+1} - a_{m+1})}{(a_{m+2-s} - a_{m+1})(a_{m+1-s} - a_{m+1})}, \qquad s = 2, \ldots, m.$$

Proof. Suppose that $\lambda + a_i$ is a factor of $P_n(t, \lambda)$. Expanding along the i-th row/column gives

$$P_n(t, -a_i) = -tc_{i-1} P_{1,i-2}(t, -a_i) P_{i+1,n}(t, -a_i) - tc_i P_{1,i-1}(t, -a_i) P_{i+2,n}(t, -a_i).$$

Thus, if $\lambda + a_i$ is a constant branch, then

$$c_{i-1} P_{1,i-2}(t, -a_i) P_{i+1,n}(t, -a_i) + c_i P_{1,i-1}(t, -a_i) P_{i+2,n}(t, -a_i) = 0.$$

If $i = 1$ or n, this is immediately impossible: sending t to 0 would force a repeated diagonal entry.

Let

$$M = \frac{P_{1,i-2}}{P_{1,i-1}}, \qquad N = \frac{P_{i+2,n}}{P_{i+1,n}},$$

evaluated at $\lambda = -a_i$. The constant branch condition becomes

$$c_{i-1}M + c_i N = 0$$

The two rational functions $M(t)$ and $N(t)$ are continued fractions of depths $i-1$ and $n-i$. By Lemma 4.2, we must have that these depths agree. Thus $i-1 = n-i$, and since $n = 2m+1$, this forces $i = m+1$. Applying Lemma 4.2 gives the displayed relations among the c_i.

Conversely, when $i = m+1$, Lemma 4.2 shows that the displayed relations are exactly the conditions under which $c_m M + c_{m+1} N = 0$. Hence, they are also sufficient for $\lambda + a_{m+1}$ to divide $P_n(t, \lambda)$. □

It is clear from the proof that we may make the following relaxation of the above theorem.

Corollary 4.4. *Suppose that $n = 2m+1$. If $\lambda + \alpha$ is a linear factor of $P_n(t, \lambda)$ and $a_i = \alpha$ for a unique index $i \in [n]$, then $i = m+1$.*

Moreover, this gives us the following regarding the dimension of constant branches in the locus where no $c_i = 0$ and the a_i are pairwise distinct.

Corollary 4.5. *When $n = 2m+1$, for a fixed pairwise distinct diagonal a, the locus of connected couplings admitting constant branches has codimension m.*

Proof. In this space the couplings $c_{m+1}, \ldots, c_{2m}$ are uniquely determined by $c_1, \ldots, c_m$. □

When multiple values of a_i are allowed to take the same value, constant branches are much easier to come by. Although we will not attempt to give a complete classification of all such mechanisms, we illustrate one additional mechanism.

Theorem 4.6. *Let $n = 2m+1$. Suppose that $a_1 = a_3 = \cdots = a_{n-2} = a_n$. Then $\lambda + a_n \mid P_n(t, \lambda)$.*

Proof. Follows immediately from the Leibniz expansion. In particular, every permutation that contributes a nonzero summand has an odd indexed fixed point. □

We now show that for every fixed connected coupling, the codimension of parameters that admit constant branches in the diagonal variables is $\lfloor n/2 \rfloor$.

Theorem 4.7. *Let $n = 2m+1$, and fix c such that $c_i \neq 0$ for all i. The codimension of the space of potentials that admit constant branches is m.*

Proof. Suppose that $\lambda + a_j$ is a constant branch. Then

$$P_n(t, -a_j) \equiv 0,$$

which gives m coefficient equations in the variable t. To prove that the codimension is exactly m, we compute the dimension of the incidence where a constant branch is allowed to occur at an unspecified value.

Let $\lambda = -\alpha$, $z_i = a_i - \alpha$, and

$$K_n(t, z) = P_n(t, -\alpha).$$

Then $K_0 = 1$, $K_1 = z_1$, and

$$K_k = z_k K_{k-1} - tc_{k-1} K_{k-2}.$$

We write $K_k(t, z, c)$ to allow us to modify the off-diagonal terms.

Define

$$X_n(c) = \{z \in \mathbb{C}^n : K_n(t, z) \equiv 0\}.$$

We first show that

$$\dim X_{2m+1}(c) = m.$$

As the constant term of K_n as a polynomial in t is $\prod z_i$, necessarily some z_i must vanish. We analyze each piece $X_n(c) \cap \{z_i = 0\}$.

If $i = 1$ or $i = n$, then

$$K_n(t, 0, z_2, \dots, z_n) = -tc_1 K_{n-2}(t, z_3, \dots, z_n, c_3, \dots, c_{n-1})$$

or

$$K_n(t, z_1, \dots, z_{n-1}, 0) = -tc_{n-1} K_{n-2}(t, z_1, \dots, z_{n-2}, c_1, \dots, c_{n-3}),$$

respectively. Thus, $X_n(c) \cap \{z_1 = 0\}$ and $X_n(c) \cap \{z_n = 0\}$ are each given by the product of a free coordinate and their corresponding $X_{n-2}(c')$.

If $2 \le i \le n-1$, then setting $z_i = 0$ contracts the two neighboring vertices into the linear combination $c_i z_{i-1} + c_{i-1} z_{i+1}$. More precisely,

$$\begin{aligned} &K_n(t, z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_n) \\ &\qquad = -t\, K_{n-2}\big(t, z_1, \dots, z_{i-2}, c_i z_{i-1} + c_{i-1} z_{i+1}, z_{i+2}, \dots, z_n; c'\big), \end{aligned}$$

for the induced contracted coupling vector c'.

Note that if $i = 2$, then the left coupling in the contracted chain does not appear, and if $i = n-1$, then the right coupling does not appear.

This identity follows by a straightforward computation that we include for the reader's convenience: Recall that

$$K_{1,n}|_{z_i=0} = -t(c_{i-1} K_{1,i-2} K_{i+1,n} + c_i K_{1,i-1} K_{i+2,n}).$$

Expand

$$K_{i+1,n} = z_{i+1} K_{i+2,n} - tc_{i+1} K_{i+3,n}, \qquad K_{1,i-1} = z_{i-1} K_{1,i-2} - tc_{i-2} K_{1,i-3}.$$

Substituting gives the contracted continuant displayed above, which is exactly the determinant of the Jacobi matrix described. Thus, each entry of c' is either an original coupling or a product of two original couplings; that is, c' is connected.

Equivalently, on the hyperplane $\{z_i = 0\}$, there is a linear map

$$\pi_i : \{z_i = 0\} \longrightarrow \mathbb{C}^{n-2}$$

whose coordinates are those of the contracted chain. This map is surjective and has one-dimensional fibers: for $2 \le i \le n-1$, the only nontrivial coordinate is $c_i z_{i-1} + c_{i-1} z_{i+1}$, and both c_{i-1} and c_i are nonzero.

Now we can prove by induction that $X_{2m+1}(c)$ has dimension m. The base case $n = 1$ is immediate.

Now assume that $n = 2m+1$. Since the constant term of K_n is $\prod z_i$, the condition $K_n(t,z) \equiv 0$ implies that some $z_i = 0$. By the work above, for each such i there is a contracted coupling vector c' and a surjective contraction map π_i, with one-dimensional fibers, such that

$$X_n(c) \cap \{z_i = 0\} = \pi_i^{-1}(X_{n-2}(c')).$$

By induction, $\dim X_{n-2}(c') = m-1$. Hence, each piece has dimension m, and so $\dim X_n(c) = m$. Thus, $X_n(c)$ has codimension $m+1$ in $\mathbb{C}^n$.

Finally, consider

$$Y = \{(\alpha, z) : z \in X_{2m+1}(c)\} \subset \mathbb{C} \times \mathbb{C}^{2m+1}.$$

It has dimension $m+1$. The map

$$Y \longrightarrow \mathbb{C}^{2m+1}, \qquad (\alpha, z) \longmapsto a = z + \alpha(1, \dots, 1)$$

has image exactly the potential vectors admitting a constant branch. Its fibers are finite, because $z \in X_{2m+1}(c)$ forces some $z_i = 0$, so for a fixed image a the possible values of α belong to the finite set $\{a_1, \dots, a_{2m+1}\}$. Therefore, the image has dimension $m+1$, and its codimension in $\mathbb{C}^{2m+1}$ is m. □

5. Strata of Pairwise Distinct Diagonal Entries

Fix pairwise-distinct diagonal entries $a_1, \dots, a_n$. We interpret $[n]$ as the vertex set of the path graph with edge i given by $(i, i+1)$. For a subset $S \subset [n]$, define its boundary edge set by

$$\partial S := \{\, i \in \{1, \dots, n-1\} : |\{i, i+1\} \cap S| = 1 \,\}.$$

Equivalently, $i \in \partial S$ if and only if exactly one endpoint of the edge $(i, i+1)$ lies in S. For a nonempty proper subset $S \subset [n]$, define

$$\ell_S(c) := \sum_{k=1}^{n-1} \frac{\mathbf{1}_{k+1 \in S} - \mathbf{1}_{k \in S}}{a_{k+1} - a_k}\, c_k,$$

where $\mathbf{1}_{i \in S} = 1$ if $i \in S$ and is 0 otherwise.

Theorem 5.1. *Assume $a_i \neq a_j$ for any $i \neq j$. If $P_n(t, \lambda)$ has a factor whose Hensel subset at $t = 0$ is S, then $\ell_S(c) = 0$.*

Proof. Write $A_i = \lambda + a_i$ and $P_n^{[r]} = [t^r]P_n$, so that

$$P_n^{[0]} = \prod_{i=1}^{n} A_i.$$

By the Leibniz expansion,

$$P_n^{[1]} = -\sum_{i=1}^{n-1} c_i \prod_{j \neq i, i+1} A_j. \tag{1}$$

Let

$$F_0 = \prod_{p \in S} A_p, \qquad G_0 = \prod_{q \notin S} A_q,$$

and suppose

$$P_n = (F_0 + tF_1 + \cdots)(G_0 + tG_1 + \cdots).$$

Notice that

$$F_1 G_0 + F_0 G_1 = P_n^{[1]}. \tag{2}$$

By Lemma 2.3, we have $\deg F_1 \leq |S| - 2$. Evaluating (2) at $\lambda = -a_p$, $p \in S$, gives

$$F_1(-a_p) = \frac{P_n^{[1]}(-a_p)}{G_0(-a_p)}.$$

The unique polynomial of degree at most $|S| - 1$ interpolating F_1 through the values $\{-a_i \mid i \in S\}$ is

$$I(\lambda) = \sum_{p \in S} F_1(-a_p) L_p(\lambda),$$

where

$$L_p(\lambda) = \prod_{q \in S \setminus \{p\}} \frac{\lambda + a_q}{a_q - a_p} = \frac{F_0(\lambda)}{(\lambda + a_p) F_0'(-a_p)}.$$

Thus,

$$I(\lambda) = \sum_{p \in S} F_1(-a_p) \frac{F_0(\lambda)}{(\lambda + a_p) F_0'(-a_p)} = \sum_{p \in S} \frac{P_n^{[1]}(-a_p)}{G_0(-a_p)} \frac{F_0(\lambda)}{(\lambda + a_p) F_0'(-a_p)}.$$

Since $\deg F_1 \leq |S| - 2$, uniqueness of interpolation gives $I = F_1$. Consequently, the coefficient of $\lambda^{|S|-1}$ in I vanishes.

As $\frac{F_0(\lambda)}{\lambda + a_p}$ is monic, $I(\lambda)$ has the coefficient of $\lambda^{|S|-1}$ given by

$$\sum_{p \in S} \frac{P_n^{[1]}(-a_p)}{F_0'(-a_p) G_0(-a_p)} = \sum_{p \in S} \frac{P_n^{[1]}(-a_p)}{(P_n^{[0]})'(-a_p)} = \sum_{p \in S} \mathrm{Res}_{\lambda = -a_p} \frac{P_n^{[1]}(\lambda)}{P_n^{[0]}(\lambda)}.$$

Here Res denotes the standard residue.

Notice that

$$\frac{P_n^{[1]}}{P_n^{[0]}} = -\sum_{i=1}^{n-1} \frac{c_i}{A_i A_{i+1}},$$

$$\operatorname{Res}_{\lambda=-a_i} \frac{c_i}{A_i A_{i+1}} = \frac{c_i}{a_{i+1} - a_i}, \qquad \operatorname{Res}_{\lambda=-a_{i+1}} \frac{c_i}{A_i A_{i+1}} = \frac{c_i}{a_i - a_{i+1}}.$$

In particular, if both i and $i+1$ are in S, or if neither are in S, then

$$\sum_{p \in S} \operatorname{Res}_{\lambda=-a_p} \left(-\frac{c_i}{A_i A_{i+1}} \right) = 0.$$

Otherwise, if $i \in S$ and $i+1 \notin S$, we have

$$\sum_{p \in S} \operatorname{Res}_{\lambda=-a_p} \left(-\frac{c_i}{A_i A_{i+1}} \right) = -\frac{c_i}{a_{i+1} - a_i},$$

whereas if $i \notin S$ and $i+1 \in S$, we have

$$\sum_{p \in S} \operatorname{Res}_{\lambda=-a_p} \left(-\frac{c_i}{A_i A_{i+1}} \right) = -\frac{c_i}{a_i - a_{i+1}}.$$

Thus, this vanishing coefficient is exactly $\ell_S(c) = 0$. □

Corollary 5.2. *If ∂S has only one element, then $\ell_S(c) = 0$ forces some $c_i = 0$. For such S, there are no corresponding factorizations in T_c. In particular, any proper Hensel set S given by an initial or terminal interval (i.e. $\{1, \ldots, r\}$ or $\{r, \ldots, n\}$) can only occur if there is a cut.*

Next we show that the collection of P_n that have a factor whose Hensel subset is S at $t = 0$ must be a proper subset of $\{\ell_S(c) = 0\}$.

Lemma 5.3. *Fix $a \in \mathbb{C}^n$ and a subset $S \subset [n]$. Set*

$$F_{0,S} = \prod_{i \in S} (\lambda + a_i), \qquad G_{0,S} = \prod_{i \notin S} (\lambda + a_i).$$

The set of coupling vectors $c \in \mathbb{C}^{n-1}$ for which there exist monic polynomials F, G satisfying

$$P_n(t, \lambda) = F(t, \lambda) G(t, \lambda), \qquad F(0, \lambda) = F_{0,S}, \qquad G(0, \lambda) = G_{0,S},$$

is Zariski closed.

Proof. As factors are necessarily monic in λ, this follows from the standard projective factor-incidence construction, in the same spirit as the closedness argument used in [FL26]. In particular, let $d = |S|$, and suppose that F and G are factors of $P_n(t, \lambda)$, where F has prescribed specialization $F_{0,S}$. That is,

$$F_0 = \prod_{i \in S} (\lambda + a_i), \qquad G_0 = \prod_{i \notin S} (\lambda + a_i).$$

By Lemma 2.3, any factor of degree d has support contained in

$$\Sigma_d = \{(r, j) : j + 2r \leq d\},$$

and the complementary factor has support contained in Σ_{n-d}.

Let V_d and V_{n-d} be the corresponding vector spaces of polynomials in t, λ, and let ℓ_F, ℓ_G denote the coefficients of λ^d and λ^{n-d}. Consider the closed incidence variety

$$\mathcal{X}_S \subset \mathbb{C}^{n-1} \times \mathbb{P}(V_d) \times \mathbb{P}(V_{n-d})$$

defined by

$$fg = \ell_F(f)\ell_G(g)P_n(t, \lambda; c),$$

$$[t^0]f = \ell_F(f)F_0, \qquad [t^0]g = \ell_G(g)G_0.$$

As the product $\mathbb{P}(V_d) \times \mathbb{P}(V_{n-d})$ is proper, so is the natural projection $\mathbb{C}^{n-1} \times \mathbb{P}(V_d) \times \mathbb{P}(V_{n-d}) \to \mathbb{C}^{n-1}$. Its restriction to $\mathcal{X}_S$ is also proper, and thus has closed image. This image is the collection of c such that P_n admits a factor F with specialization $F_{0,S}$.

Indeed, a monic factorization gives a point of $\mathcal{X}_S$. Conversely, if $(c, [f], [g]) \in \mathcal{X}_S$, then $fg \neq 0$, so $\ell_F(f)\ell_G(g) \neq 0$. Normalizing by these two nonzero leading coefficients gives monic factors F, G with

$$FG = P_n, \qquad F^{[0]} = F_0, \qquad G^{[0]} = G_0.$$

Thus, the prescribed-specialization locus is closed. □

Remark 5.4. When the a_i are pairwise distinct, the preceding locus is exactly the fixed Hensel subset locus associated with S. When diagonal values collide, the subset S need not be unique; different subsets may determine the same specialization polynomial $F_{0,S}$.

Lemma 5.5. *Assume that the a_i are pairwise distinct. Let S be a nonempty proper subset such that neither S nor S^c has size 1 and such that ∂S has at least two elements. Let $\mathcal{R}_S \subset \mathbb{C}^{n-1}$ be the affine closed locus where $P_n(t, \lambda)$ has a factor with Hensel subset S, and set $H_S = \{\ell_S = 0\}$. Then $\mathcal{R}_S \neq H_S$.*

Proof. First note that ∂S has two non-adjacent i. Indeed, otherwise ∂S would only have two i which are adjacent, and then S or S^c must be a single element.

Now choose p, q to be two non-adjacent elements of ∂S. The coefficients of c_p and c_q are then nonzero in ℓ_S.

Let us now fix nonzero values for c_p and c_q, with all other $c_i = 0$, such that $\ell_S(c) = 0$. At this cut point, $P_n(t, \lambda)$ becomes a product of

$$A_p A_{p+1} - tc_p, \qquad A_q A_{q+1} - tc_q$$

and constant linear factors. Since $c_p, c_q \neq 0$, these two quadratics are irreducible. Because S contains exactly one endpoint of each of the two chosen boundary edges, no factor at this cut point can have Hensel subset S: such a factor would split both irreducible quadratics. Thus, this point lies in $H_S \setminus \mathcal{R}_S$. Therefore $\mathcal{R}_S \neq H_S$. $\square$

Theorem 5.6. *Fix $n \geq 4$, and pairwise distinct diagonal entries. Then*

$$\mathrm{codim}_{T_c}\{c \in T_c : P_n(t, \lambda) \text{ is reducible}\} \geq 2.$$

Equivalently, over the pairwise-distinct diagonal locus, the only codimension 1 components of reducible parameters are the hyperplanes $c_i = 0$.

Proof. By Theorem 5.1, for each proper Hensel subset S, the corresponding collection of c such that P_n has a factor with Hensel subset S is contained in $\{\ell_S = 0\}$. If S has only one boundary edge, this forces some $c_i = 0$ and so this stratum misses T_c.

If S or S^c has size one, the factorization has a constant branch factor. In even length this is impossible on the connected pairwise-distinct stratum by Corollary 3.2. In odd length, the collection of parameters with constant branch factors has codimension $\lfloor n/2 \rfloor$ on this stratum by Corollary 4.5. For $n \geq 4$, this is at least codimension 2 in T_c.

Finally, all remaining S satisfy the hypotheses of Lemma 5.5. Let $\mathcal{R}_S \subset \mathbb{C}^{n-1}$ be the affine closed fixed-S locus from Lemma 5.3, and set $H_S = \{\ell_S = 0\}$. By Theorem 5.1,

$$\mathcal{R}_S \subseteq H_S.$$

The polynomial ℓ_S is a nonzero linear form in the coupling variables, hence H_S is an irreducible affine hyperplane. Since ℓ_S has at least two nonzero coordinate terms, H_S is not contained in any coordinate hyperplane, and therefore $H_S \cap T_c$ is dense in H_S. Lemma 5.5 gives $\mathcal{R}_S \neq H_S$. Therefore $\mathcal{R}_S$ is a proper closed subset of the irreducible hyperplane H_S, and so it has codimension at least 2 in $\mathbb{C}^{n-1}$. Intersecting with the open torus T_c preserves this codimension bound.

As the reducibility locus is a finite union of all possible Hensel sets which are at least codimension 2 in T_c, we conclude that the locus itself is at least codimension 2 in T_c. $\square$

6. Generic Irreducibility for Fixed Diagonal Data

In this section we prove the following.

Theorem 6.1. (1) *Let $n = 2m + 1 \geq 3$ and a be fixed. For generic $c \in T_c$, P_n is reducible if and only if $a_1 = a_3 = \cdots = a_{2m+1}$.*

(2) *Let $n = 2m \geq 4$ and a be fixed. For generic $c \in T_c$, P_n is reducible if and only if $a_1 = a_3 = \cdots = a_{2m-1}$ and $a_2 = a_4 = \cdots = a_{2m}$.*

First we prove the right-hand side implies the left-hand side.

Lemma 6.2. (1) *Assume $n = 2m + 1 \geq 3$. For fixed a and generic $c \in T_c$, P_n is reducible if*

$$a_1 = a_3 = \cdots = a_{2m+1}.$$

(2) *Assume $n = 2m \geq 4$. For fixed a and generic $c \in T_c$, P_n is reducible if*

$$a_1 = a_3 = \cdots = a_{2m-1} \qquad \text{and} \qquad a_2 = a_4 = \cdots = a_{2m}.$$

Proof. Notice (1) follows from Theorem 4.6.

For (2), first suppose that $a_{2r-1} = \alpha$ and $a_{2r} = \beta$ for each $r = 1, \ldots, m$, and let $A = \lambda + \alpha$ and $B = \lambda + \beta$.

By reordering the vertices in the Jacobi matrix by listing the odd vertices and then the even vertices, we have

$$J_n(w) + \lambda I = \begin{pmatrix} AI_m & wD \\ wD^T & BI_m \end{pmatrix},$$

where D is the weighted odd-even incidence matrix. Taking the Schur complement gives

$$\chi_n(w, \lambda) = \det(ABI_m - w^2 D^T D),$$

and hence, in the reduced variable $t = w^2$,

$$P_n(t, \lambda) = \det(ABI_m - tD^T D).$$

If the eigenvalues of $D^T D$ are $\rho_1, \ldots, \rho_m$, then

$$P_{2m}(t, \lambda) = \prod_{i=1}^{m} (AB - \rho_i t).$$

□

To prove the remaining direction of Theorem 6.1, we need the following lemma.

Lemma 6.3. *Let $n \geq 2$. Assume that $P_n(t, \lambda)$ is irreducible for fixed couplings $(c_1, \ldots, c_{n-1}) \in \mathbb{C}^{n-1}$, and assume that*

$$P_{n-1}(t, -a_{n+1}) \not\equiv 0.$$

Then, $P_{n+1}(t, \lambda)$ is irreducible for generic $c_n \in \mathbb{C}$.

Proof. Suppose that P_{n+1} is reducible for a generic choice of c_n, and let $A = \lambda + a_{n+1}$. As in Lemma 5.3, for $d = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$ form the projective incidence variety of degree d and degree $n + 1 - d$ monic factor pairs, using Lemma 2.3 to keep the coefficient spaces finite-dimensional. For each d, the image in $\mathbb{C}$ is closed. As P_{n+1} is generically reducible, this means that the union of these finitely many varieties must be the entire $\mathbb{C}$. In particular, for some d the image of the corresponding projective incidence variety is all of $\mathbb{C}$.

At $c_n = 0$, we have

$$P_{n+1} = AP_n.$$

By assumption, P_n is irreducible and A is coprime to P_n. Thus, d must be 1. Hence P_{n+1} has a linear factor for all $c_n \in \mathbb{C}$.

By Lemma 2.3, any monic linear factor is independent of t. At $t = 0$, this linear factor divides the fixed polynomial

$$\prod_{i=1}^{n+1} (\lambda + a_i),$$

so its root must be one of the finitely many values $-a_i$. Note that for each a_i, the locus of $c_n \in \mathbb{C}$ such that $\lambda + a_i$ is a factor of P_{n+1} is algebraic. Thus, at least one of these linear factors must always be a factor of P_{n+1}. Specializing at $c_n = 0$, the only possible linear factor is A. Hence, the generic linear factor is identically A, and A would divide P_{n+1} for generic c_n. But

$$P_{n+1}(t, -a_{n+1}) = -tc_n P_{n-1}(t, -a_{n+1}),$$

which is not identically zero by assumption, giving us a contradiction. □

6.1. **Proof of Theorem 6.1.**

Proof. We prove the claim by induction.

The base cases are clear. When $n = 2$, then

$$P_2 = (\lambda + a_1)(\lambda + a_2) - tc_1,$$

which is irreducible in $\mathbb{C}[t, \lambda]$ for all $c_1 \neq 0$. When $n = 3$, then

$$P_3 = (\lambda + a_1)(\lambda + a_2)(\lambda + a_3) - t(c_1(\lambda + a_3) + c_2(\lambda + a_1)).$$

If $a_1 = a_3$, then $\lambda + a_1$ is a factor. Conversely, a reducible cubic must have a linear factor, and by Lemma 2.3 this factor is independent of t. Testing the possible roots $-a_1, -a_2, -a_3$ shows that, for generic nonzero c_1, c_2, this can only happen when $a_1 = a_3$.

Now consider $n \geq 4$. Let $n = 2m$ and suppose that the diagonal is not constant for at least one parity class.

If the odd class is nonconstant, then P_{2m-1} is such that its odd entries are nonconstant, and thus it is irreducible for a generic coupling by induction. Moreover, $P_{2m-2}(t, -a_{2m}) \not\equiv 0$, since an even-length continuant has a nonzero top matching coefficient independent of λ. Thus, all conditions of Lemma 6.3 are met, and we conclude that P_n is generically irreducible.

If instead the even class is nonconstant, just repeat the same argument on the reverse Jacobi matrix with labeling $(a_n, \dots, a_1)$, $(c_{n-1}, \dots, c_1)$.

Now suppose that $n = 2m + 1$ and the odd diagonal class $a_1, a_3, \ldots, a_{2m+1}$ is not constant. At least one of the two even blocks

$$P_{1,2m}, \qquad P_{2,2m+1}$$

is not parity-constant in the sense of the even case. Indeed, if both were parity-constant, then the odd entries of $P_{1,2m}$ would make $a_1 = a_3 = \cdots = a_{2m-1}$, and the even entries of $P_{2,2m+1}$ would make $a_3 = a_5 = \cdots = a_{2m+1}$, so all odd entries of the original chain would be equal.

First suppose $P_{1,2m}$ is not parity-constant. By the even case of the induction, $P_{1,2m}$ is irreducible for a generic choice of $c_1, \ldots, c_{2m-1}$. We claim that

$$P_{1,2m-1}(t, -a_{2m+1}) \not\equiv 0$$

as a polynomial in t and in the couplings $c_1, \ldots, c_{2m-2}$. The coefficient of t^{m-1} is a sum over maximum matchings of the path $1, \ldots, 2m-1$. These maximum matchings leave exactly one odd vertex $2r-1$ unmatched. The corresponding coupling monomials are distinct, and after substituting $\lambda = -a_{2m+1}$, their coefficients are $a_{2r-1} - a_{2m+1}$, up to a common sign. Hence, the coefficient can vanish identically only if

$$a_1 = a_3 = \cdots = a_{2m-1} = a_{2m+1},$$

contrary to the assumption that the odd class is not constant. Thus, Lemma 6.3, applied to the prefix $P_{1,2m}$ and the terminal vertex $2m+1$, implies that P_n is generically irreducible.

If $P_{1,2m}$ is parity-constant, then $P_{2,2m+1}$ is not parity-constant. Apply the same argument to the reversed chain

$$(a_{2m+1}, a_{2m}, \ldots, a_1), \qquad (c_{2m}, c_{2m-1}, \ldots, c_1).$$

This proves generic irreducibility in the remaining odd case. $\square$

Corollary 6.4. *Let*

$$\chi_n(w, \lambda) = P_n(w^2, \lambda).$$

For $n \geq 3$, P_n and χ_n have the same generic irreducibility classification for every fixed diagonal vector.

More explicitly, if $n = 2m + 1$, then χ_n is generically reducible if and only if

$$a_1 = a_3 = \cdots = a_{2m+1}.$$

If $n = 2m \geq 4$, then χ_n is generically reducible if and only if

$$a_1 = a_3 = \cdots = a_{2m-1}, \qquad a_2 = a_4 = \cdots = a_{2m}.$$

Proof. For a fixed diagonal vector, suppose P_n is generically irreducible but $\chi_n(w, \lambda) = P_n(w^2, \lambda)$ is generically reducible. By Lemma 2.5, this can only happen when $n = 2m$, and then χ_n has a balanced $m + m$ factorization

$$\chi_n(w, \lambda) = F(w, \lambda)F(-w, \lambda).$$

The parity two-color diagonal locus is given by

$$a_1 = a_3 = \cdots = a_{2m-1}, \qquad a_2 = a_4 = \cdots = a_{2m}.$$

As P_n is generically irreducible, our fixed diagonal vector must be away from this locus, and thus at least one parity class is nonconstant.

If the odd class is nonconstant, then by the odd case of Theorem 6.1 and Corollary 2.6, the prefix $\chi_{1,2m-1}$ is generically irreducible. Fix generic values of the prefix couplings for which $\chi_{1,2m-1}$ is irreducible. By the same argument as given in Lemma 5.3, but in $\mathbb{C}[w, \lambda]$, a generic balanced $m + m$ factorization would persist after specializing $c_{2m-1} = 0$. But then

$$\chi_{1,2m} = (\lambda + a_{2m})\chi_{1,2m-1},$$

whose only proper factor degrees are 1 and $2m-1$, not m. This is a contradiction. If the even class is nonconstant, the same argument applies after cutting $c_1 = 0$ and using the suffix chain. $\square$

7. Dimension Bound on the Reducibility Locus

We are now ready to prove our main result, a lower bound on the codimension of the reducibility locus outside of the cut hyperplanes.

Theorem 7.1. *Let*

$$\mathcal{U}_n = \mathbb{C}^n \times T_c.$$

For $n \geq 4$, the reduced reducible locus

$$R_n^P = \{(a, c) \in \mathcal{U}_n : P_n(t, \lambda) \text{ is reducible in } \mathbb{C}[t, \lambda]\}$$

has codimension at least 2. *Moreover, the original reducible locus*

$$R_n^\chi = \{(a, c) \in \mathcal{U}_n : \chi_n(w, \lambda) \text{ is reducible in } \mathbb{C}[w, \lambda]\}$$

also has codimension at least 2.

Proof. Stratify $\mathbb{C}^n$ by equality patterns of the diagonal entries. By the same projective factor-incidence argument as in Lemma 5.3, both reducible loci are closed. For the continuant, Theorem 5.6 gives us codimension at least 2 over the pairwise-distinct diagonal stratum.

Now let D_π be a diagonal equality stratum. If $\mathrm{codim} D_\pi \geq 2$, then even if the whole family over D_π were reducible, it would already have codimension at least 2 in $\mathcal{U}_n$.

Thus, only diagonal hypersurfaces of the form $a_p = a_q$ could produce divisors. By Theorem 6.1, generic reducibility over a fixed diagonal vector requires the relevant parity classes to be constant: the odd class in odd length, and both parity classes in even length. For $n \geq 4$, no single equality hyperplane $a_p = a_q$ forces these parity-constant conditions; those conditions have codimension at least 2 in the diagonal space. Hence generic connected couplings over any single diagonal equality hyperplane give an irreducible continuant. Therefore the reducible locus inside each such hyperplane is a proper closed subset, and thus has codimension at least 2 in $\mathcal{U}_n$. Hence

$$\operatorname{codim}_{\mathcal{U}_n} R_n^P \geq 2.$$

On the pairwise distinct stratum, Lemma 2.4 identifies the P_n and χ_n reducible locus. On each equality stratum, Corollary 6.4 allows the preceding argument to be repeated for χ_n. □

Acknowledgments

This research was partially supported by NSF grant DMS-2052519.

Statements and Declarations

Conflict of Interest The author declares no conflicts of interest.

Data Availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

[FL26] Matthew Faust and Wencai Liu. Generic irreducibility of Bloch varieties for periodic graph operators, 2026.

[FLG25] M. Faust and J. Lopez Garcia. Irreducibility of the dispersion polynomial for periodic graphs. *SIAM J. Appl. Algebra Geom.*, 9(1):83–107, 2025.

[FLM22] Jake Fillman, Wencai Liu, and Rodrigo Matos. Irreducibility of the Bloch variety for finite-range Schrödinger operators. *J. Funct. Anal.*, 283(10):Paper No. 109670, 22, 2022.

[Liu22] Wencai Liu. Irreducibility of the Fermi variety for discrete periodic Schrödinger operators and embedded eigenvalues. *Geom. Funct. Anal.*, 32(1):1–30, 2022.

[Sha26a] B. Shapiro. Reducibility of spectral curves of finite Jacobi pencils, 2026.

[Sha26b] B. Shapiro. Reducibility of spectral curves of finite Jacobi pencils (version 1), 2026.

Matthew Faust, Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

Email address: `mfaust@msu.edu`

URL: `https://mattfaust.github.io/`