Floquet Theory for Discrete Periodic Operators

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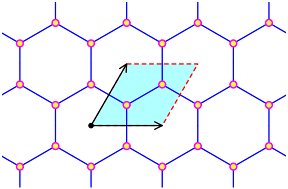
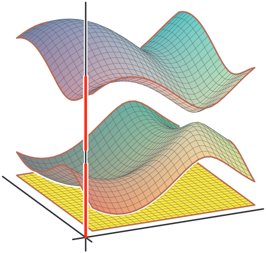




Discrete Periodic Schrödinger Operator

Let $V : \mathbb{Z}^d \to \mathbb{R}$ be a periodic function.

Schrödinger Operator

A discrete periodic Schrödinger operator $H = V + \Delta$ acts on functions $f : \mathbb{Z}^d \to \mathbb{C}$ as follows, for all $n \in \mathbb{Z}^d$:

$$Hf(n) = V(n)f(n) + \sum_{|n-m|=1} f(m).$$

Let $q = (q_1, \ldots, q_d) \in \mathbb{Z}^d$, and let $q\mathbb{Z} = q_1\mathbb{Z} \times \cdots \times q_d\mathbb{Z}$.
$V : \mathbb{Z}^d \to \mathbb{R}$ is a $q\mathbb{Z}$-periodic potential if

$$V(n + q_i e_i) = V(n)$$

for each $i = 1, \ldots, d$, where $q_i e_i = (0, \ldots, 0, q_i, 0, \ldots, 0)$.

Discrete Periodic Schrödinger Operator Examples

Schrödinger Operator

A discrete periodic Schrödinger operator $H = V + \Delta$ acts on functions $f : \mathbb{Z}^d \to \mathbb{C}$ as follows, for all $n \in \mathbb{Z}^d$:

$$Hf(n) = V(n)f(n) + \sum_{|n-m|=1} f(m).$$

For example, when $d = 1$, we get the classic operator

$$Hf(n) = V(n)f(n) + f(n-1) + f(n+1).$$

When $d = 2$ and $n = (n_1, n_2)$:

$$Hf(n) = V(n)f(n) + f(n_1 - 1, n_2) + f(n_1 + 1, n_2)$$

$$+f(n_1, n_2 - 1) + f(n_1, n_2 + 1).$$

The Spectrum: Case of $d = 1$, $V(n)$ $\mathbb{Z}$-periodic.

We wish to study the spectrum $\sigma(H)$ of $H = V + \Delta$ on $\ell^2(\mathbb{Z}^d)$. Let us consider the case when $d = 1$ and $V(n) = V$.

Let $F : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ be the Fourier transform

$$f(n) \to \hat{f}(z) = \sum_{a \in \mathbb{Z}} f(n+a) z^{-a}.$$

By Parseval's identity, F is unitary, and so $\sigma(H) = \sigma(FHF^{-1})$. It is easy to see that $F(f(n+a)) = z^a F(f(n))$. Thus for $z \in \mathbb{T}$,

$$(FHF^{-1}(\hat{f}))(z) = V\hat{f}(z) + z^{-1}\hat{f}(z) + z\hat{f}(z).$$

It is easy to show that $\lambda \in \sigma(FHF^{-1})$ when $\lambda = V + z^{-1} + z$ for some $z \in \mathbb{T}$.

Equivalently, $\sigma(H) = \{V + 2\cos(k) \mid k \in \mathbb{R}\} = [V-2, V+2]$.

Floquet Transform for a $q\mathbb{Z}$-periodic Potential

Let $Q = \prod q_i$, and consider an arbitrary d and a $q\mathbb{Z}$-periodic potential $V(n)$ on $\mathbb{Z}^d$. Define

$$W = \{\omega \in \mathbb{Z}^d : \omega_i \in \{0, \ldots, q_i - 1\}\},$$

so each $\omega \in W$ labels a unique orbit $\omega + q\mathbb{Z}$ in $\mathbb{Z}^d / q\mathbb{Z}$.
The Floquet transform

$$\mathcal{F} \colon \ell^2(\mathbb{Z}^d) \;\to\; \left(L^2(\mathbb{T}^d)\right)^W$$

is given by

$$f(n) \;\mapsto\; \hat{f}_n(z) \;=\; \sum_{a \in \mathbb{Z}^d} f\left(n + a\,q\right) z^{-a},$$

where $z^a = z_1^{a_1} \cdots z_d^{a_d}$ and $a\,q = (a_1 q_1, \ldots, a_d q_d)$. Similar to the standard discrete Fourier transform, $\mathcal{F}$ is unitary.

Floquet Transform for a $q\mathbb{Z}$-periodic Potential. Part II

Recall: $W = \{\omega \in \mathbb{Z}^d : \omega_i \in \{0, \ldots, q_i - 1\}\}$.
For any $n \in \mathbb{Z}^d$, $n = \omega + aq$ where $\omega \in W$ and $a \in \mathbb{Z}^d$, we have

$$\hat{f}_n(z) \;=\; \hat{f}_{\omega+aq}(z) \;=\; z^a\, \hat{f}_\omega(z), \quad \text{where} \quad \hat{f}_\omega(z) \;=\; \sum_{g\in\mathbb{Z}^d} f\left(\omega{+}g\,q\right) z^{-g}.$$

Hence, $\mathcal{F}$ acts as a Fourier transform *on each orbit* $\omega + q\mathbb{Z}$, $\omega \in W$.
For $z \in \mathbb{T}^d$, the transformed function $\hat{f}(z)$ can be viewed as a Q-dimensional vector in $\left(L^2(\mathbb{T}^d)\right)^W$:

$$\hat{f}(z) \;=\; \left(\hat{f}_\omega(z)\right)_{\omega\in W}.$$

Floquet Transform: $q\mathbb{Z}$-periodic Potential. Part III

$W = \{\omega \in \mathbb{Z}^d \mid \omega_i \in \{0, \dots, q_i - 1\}\}$, $\hat{f}(z) = \left(\hat{f}_\omega(z)\right)_{\omega \in W}$.

Let $\hat{H} := \mathcal{F} H \mathcal{F}^{-1}$. For $z \in \mathbb{T}^d$ and $\omega \in W$, as each $n = v + aq$ for some $v \in W$ and $a \in \mathbb{Z}^d$ and $\hat{f}_{v+aq}(z) \;=\; z^a \hat{f}_v(z)$,

$$\hat{H}\hat{f}_\omega(z) = V(\omega)\hat{f}_\omega(z) + \sum_{|v+aq-\omega|=1, v\in W} z^a \hat{f}_v(z).$$

So $\hat{H}$ sends $\hat{f}_\omega(z)$ to a linear combination of the coordinates of $\hat{f}(z)$ with coefficients in $\mathbb{R}[z_1^\pm, \dots, z_d^\pm]$.

Thus, there exists a matrix $Q \times Q$ matrix $H(z)$ such that

$$\hat{H}\hat{f}(z) = H(z)\hat{f}(z) \implies \hat{H} = \int_{\mathbb{T}^d}^{\oplus} H(z)\, dz,$$

and so $\sigma(H) = \sigma(\hat{H}) = \{\sigma(H(z)) \mid z \in \mathbb{T}^d\}$.

Let $W = \{(0,0),(0,1),(0,2)\}$. Remark that $\hat{f}_{\omega+aq}(z) = z^a \hat{f}_\omega(z)$ and $H(z)\hat{f}_\omega(z) = V(\omega)\hat{f}_\omega(z) + \sum_{|v-\omega|=1} \hat{f}_v(z)$.

We find that $H(z)$ acts on $\hat{f}(z)$ as follows:

$H(z)\hat{f}_{(0,0)}(z) = V(0,0)\hat{f}_{(0,0)}(z) + \hat{f}_{(-1,0)}(z) + \hat{f}_{(1,0)}(z) + \hat{f}_{(0,-1)}(z) + \hat{f}_{(0,1)}(z)$

$H(z)\hat{f}_{(0,0)}(z) = (V(0,0) + z_1^{-1} + z_1)\hat{f}_{(0,0)}(z) + z_2^{-1}\hat{f}_{(0,2)}(z) + \hat{f}_{(0,1)}(z)$,

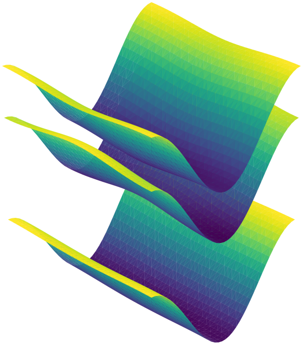
$H(z)\hat{f}_{(0,1)}(z)$ and $H(z)\hat{f}_{(0,2)}(z)$ can be obtained in a similar way.

We find that,

$$H(z) = \begin{pmatrix} V(0,0) + z_1^{-1} + z_1 & 1 & z_2^{-1} \\ 1 & V(0,1) + z_1^{-1} + z_1 & 1 \\ z_2 & 1 & V(0,2) + z_1^{-1} + z_1 \end{pmatrix}$$

Given the matrix $H(z)$, we let $D(z,\lambda) := \det(H(z) - \lambda I)$.

The Dispersion Relation



The *Dispersion relation* is $\{(z, \lambda) \in (\mathbb{T})^d \times \mathbb{R} \mid D(z, \lambda) = 0\}$.

The Fermi variety at λ_0 is given by $z \in \mathbb{T}^d$ such that $D(z, \lambda_0) = 0$.
Let $\lambda_i(z)$ be the ith smallest eigenvalue of $H(z)$.

When $z \in \mathbb{T}$, $H(z)$ is Hermitian and thus has real spectrum. Algebraically, we can study the extrema of the band functions, reducibility of $D(z, \lambda)$ (and, $D(z, \lambda_0)$), and more.

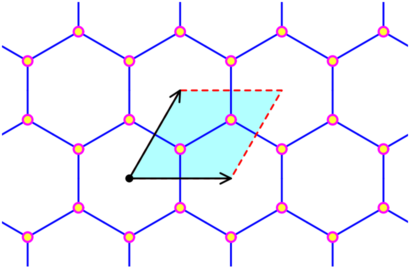
In many cases, these algebraic properties have spectral consequences. E.g., the dispersion relation is always irreducible and so $\sigma(H)$ has no pure point spectrum.

Past Work on H

1. Gieseker, Knörrer, and Trubowitz (1993 Academic Press) showed that the Fermi varieties are always irreducible for a $p\mathbb{Z} \times q\mathbb{Z}$-periodic potentials.
2. Kuchment and Vainberg (2000 Comm. PDE) showed that irreducibility of the Fermi varieties implies the absence of embedded eigenvalues.
3. Filonov and Kachkovskiy (2018 Acta Math.) there exist $q \in \mathbb{Z}^d$ for which the band edges are degenerate for all $q\mathbb{Z}$-periodic potentials.
4. Liu (2022: Geom. Funct. Anal.) generalized the results of GKT this to higher dimensions.
5. Liu (J. Anal. Math. 2022) showed that irreducibility of the dispersion relation is a key component in proving quantum ergodicity.
6. Filonov and Kachkovskiy (2024 Comm. Math. Phy.) expanded upon the 2018 paper providing more examples as well as proved for most $q \in \mathbb{Z}^d$ the band edges are non-degenerate.
7. Much more.. (e.g. Kappeler's isospectrality results).

Definition

A *$\mathbb{Z}^d$-periodic graph* Γ is a graph equipped with a free action of $\mathbb{Z}^d$ with finitely many orbits on its vertices $\mathcal{V}(\Gamma)$ and on its edges $\mathcal{E}(\Gamma)$.



Let $W \subset \mathcal{V}(\Gamma)$ contain a single vertex from each orbit of $\mathcal{V}(\Gamma)/\mathbb{Z}^d$. We call W a *fundamental domain*.

Discrete Periodic operators:

A label $c = (V, E)$ is a pair of $\mathbb{Z}^d$-periodic functions:
$V : \mathcal{V}(\Gamma) \to \mathbb{R}$, $E : \mathcal{E}(\Gamma) \to \mathbb{R}$.

$g : \mathcal{V}(\Gamma) \to \mathbb{C}$.

Discrete periodic operator:

$$(Lg)(u) := V(u)g(u) \; + \sum_{(u,v)\in\mathcal{E}(\Gamma)} E((u,v))(g(v)).$$

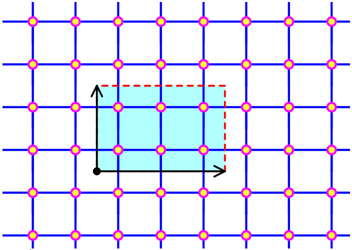
We wish to study the spectrum of L on $\ell^2(\mathcal{V}(\Gamma))$.

The discrete periodic Schrödinger operator

When $E(e) = 1$ for all $e \in \mathcal{E}(\Gamma)$, as

$$(Lg)(u) := V(u)g(u) \; + \sum_{(u,v)\in\mathcal{E}(\Gamma)} g(v).$$

We have that when Γ is the grid graph, $H = L$.



$W = \{(\omega_1, \omega_2) \in \mathbb{Z}^d \mid \omega_1 \in \{0,1,2\}, \omega_2 \in \{0,1\}\}.$

The General Floquet Matrix

Floquet theory reveals that the generalized eigenfunctions of L are quasi-periodic functions with Floquet multiplier z, for each z in $\mathbb{T}^d$:

$$g(u+a) = z^a g(u) \text{ for all } a \in \mathbb{Z}^d,\ u \in \mathcal{V}(\Gamma).$$

Such g are determined by their values on W, and so we can represent each as a finite vector $\{g(u)\}_{u \in W}$.

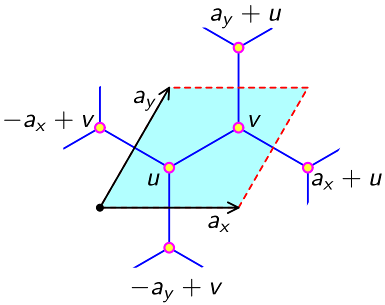
$$(Lg)(u) = V(u)g(u) + \sum_{e=(u,v+a)\in\mathcal{E}(\Gamma), v\in W} E(e)z^a g(v).$$

Thus, L acts on $\{g(u)\}_{u \in W}$ as multiplication by the $W \times W$ matrix

$$L(z)_{u,v} = \text{ coefficient of } g(v) \text{ in } (Lg)(u).$$

Collecting the eigenvalues of $L(z)$ over $z \in \mathbb{T}^d$ yields the spectrum.

Example: Hexagonal Lattice



Let g be a generalized eigenfunction of L with multiplier $(x, y) \in \mathbb{T}^2$, and let $E = 1$.

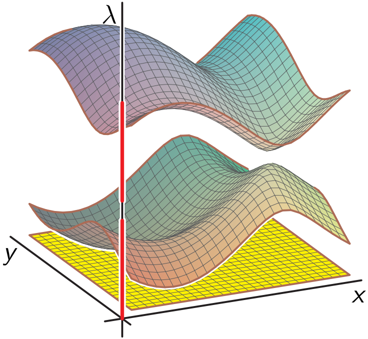
$$Lg(u) = V(u)g(u)$$
$$(1 + x^{-1} + y^{-1})g(v)$$

$$Lg(v) = (1 + x + y)g(u)$$
$$+V(v)g(v)$$

Collecting coefficients, we obtain the Floquet matrix:

$$L(x, y) = \begin{pmatrix} V(u) & -1 - x^{-1} - y^{-1} \\ -1 - x - y & V(v) \end{pmatrix}$$

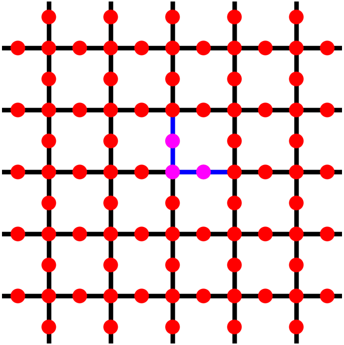
We let $D(z, \lambda) = \det(L(x, y) - \lambda I)$.



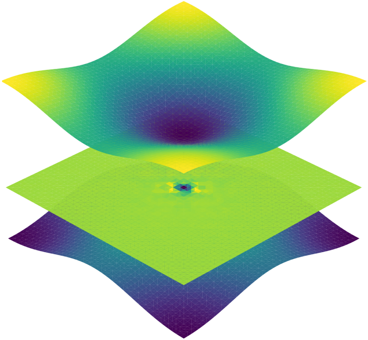
As before, the *dispersion relation* is the vanishing set of $D(z, \lambda)$ on $(\mathbb{T}^\times)^d \times \mathbb{R}$.

As we study more general models, more exotic spectral phenomena can occur.

Example: Flat Bands in the Lieb Lattice



The Lieb lattice.



The dispersion relation of the Lieb lattice when $V = 0$.

Some Past Work on Discrete Periodic Operators

1. Do, Kuchment, Sottile (J. Math. Phys. 2020): either non-degeneracy or degeneracy of the band edges occurs generically. Proved the nondegeneracy conjecture for a class of graphs.
2. Fillman, Liu, Matos (J. Func. Anal. 2022) studied the reducibility of the dispersion relation for long-range operators. Concluded irreducibility for all potentials for many models.
3. Sabri and Youssef (J. Math Phys. 2023), provided a class of graphs which can exhibit flat bands.
4. F. and Sottile (J. Spectr. Theory 2024) provided a bound on the number of isolated extrema of the band edges for any graph. Used this to extend the results of DKS to many more related models.
5. Fillman, Liu, Matos (J. Func. Anal. 2024) provided an effective criteria to prove generic irreducibility of the Fermi varieties of many graphical models.
6. Much more...

Recent Result:

Theorem F.-Liu (TBA 2025)

If V and E live outside of a particular measure 0 set, then the dispersion relation of a discrete periodic operator has no flat bands.

Related Surveys and Books:

1. Peters, Algebraic Fermi curves, Astérisque, Séminaire Bourbaki, 1990.
2. Gieseker, Knörrer, and Trubowitz, The Geometry of Algebraic Fermi Curves, Academic Press, Inc, 1993
3. Kuchment, An overview of periodic elliptic operators, Bull. Amer. Math. Soc. (2016)
4. Liu, Topics on Fermi varieties of discrete periodic Schrödinger operators, J. Math. Phys. (2022).
5. Damanik and Fillman, One-dimensional Ergodic Schrödinger Operators—II. Specific Classes, Grad. Stud. Math. AMS, 2024
6. Kuchment, Analytic and algebraic properties of dispersion relations (Bloch varieties) and Fermi surfaces. What is known and unknown, J. Math. Phys. (2023).
7. Shipman and Sottile, Algebraic aspects of periodic graph operators, arxiv preprint, (2025).

End

Thank you for listening.